

COSUPPORT COMPUTATIONS FOR FINITELY GENERATED MODULES OVER COMMUTATIVE NOETHERIAN RINGS

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ABSTRACT. We show that the cosupport of a commutative noetherian ring is precisely the set of primes appearing in a minimal pure-injective resolution of the ring. As an application of this, we prove that every countable commutative noetherian ring has full cosupport. We also settle the comparison of cosupport and support of finitely generated modules over any commutative noetherian ring of finite Krull dimension. Finally, we give an example showing that the cosupport of a finitely generated module need not be a closed subset of $\text{Spec}(R)$, providing a negative answer to a question of Sather-Wagstaff and Wickles [SWW17].

INTRODUCTION

The theory of cosupport, recently developed by Benson, Iyengar, and Krause [BIK12] in the context of triangulated categories, was partially motivated by work of Neeman [Nee11], who classified the colocalizing subcategories of the derived category of a commutative noetherian ring. Despite the many ways in which cosupport is dual to the more established notion of support introduced by Foxby [Fox79, BIK08], cosupport seems to be more elusive, even in the setting of a commutative noetherian ring. Indeed, the supply of finitely generated modules for which cosupport computations exist is limited. One purpose of this paper is to provide such computations.

We first show that for a finitely generated module over a commutative noetherian ring of finite Krull dimension, its cosupport is the intersection of its support and the cosupport of the ring, which places emphasis on computing the cosupport of the ring itself. With this in mind, we prove that countable commutative noetherian rings have full cosupport, and hence cosupport and support coincide for finitely generated modules over such rings. We also give new examples of uncountable rings that have full cosupport. Finally, we present an example of a ring whose cosupport is not closed, unlike support, yielding a negative answer to a question posed by Sather-Wagstaff and Wickles [SWW17].

One method to extract the support of a module is to identify primes appearing in its minimal injective resolution, as done by Foxby [Fox79], who used the decomposition of injective modules described by Matlis [Mat58]. Our systematic approach to computing cosupport is to appeal to the parallel decomposition of cotorsion flat modules due to Enochs [Eno84] and use minimal cotorsion flat resolutions studied in [Tho17].

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Our goal is to better understand cosupport in the setting of a commutative noetherian ring. Over such a ring R , the cosupport of a complex M is denoted $\text{cosupp}_R M$. This is the set of primes \mathfrak{p} such that $\mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, \mathbf{L}\Lambda^{\mathfrak{p}} M)$ is not acyclic, where $\mathbf{L}\Lambda^{\mathfrak{p}}(-)$ is left derived completion; see Section 1 for details. Prompted by the fact that if M is a finitely generated \mathbb{Z} -module, then $\text{cosupp}_{\mathbb{Z}} M = \text{supp}_{\mathbb{Z}} M$ [BIK12, Proposition 4.18], we investigate to what extent cosupport and support agree for finitely generated modules. The cosupport of finitely generated modules over a 1-dimensional domain having a dualizing complex is known [SWW17, Theorem 6.11]; this is recovered by part (2) of the following. Part (3) gives an affirmative answer to a question in [SWW17].

Theorem 1 (cf. Theorem 4.11, Corollary 4.12). *Let R be one of the following:*

- (1) *A countable commutative noetherian ring;*
- (2) *A finite ring extension of a 1-dimensional commutative noetherian domain that is not complete local;*
- (3) *A finite ring extension of $k[x, y]$ for any field k .*

Then R has full cosupport, i.e., $\text{cosupp}_R(R) = \text{Spec}(R)$. If $M \in \mathbf{D}(R)$ has degree-wise finitely generated cohomology, then $\text{cosupp}_R M = \text{supp}_R M$.

An obstruction to having full cosupport is completeness at a non-zero ideal. In particular, if (R, \mathfrak{m}) is a complete local ring, then $\text{cosupp}_R R = \{\mathfrak{m}\}$. Setting \mathfrak{c}_R to be the largest ideal of R such that R is \mathfrak{c}_R -complete, the inclusion

$$(\star) \quad \text{cosupp}_R R \subseteq \mathcal{V}(\mathfrak{c}_R)$$

always holds [BIK12, Proposition 4.19] (here, $\mathcal{V}(\mathfrak{c}_R) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{c}_R\}$). In Proposition 4.10, we give criteria for equality to hold in (\star) :

Proposition 2 (cf. Proposition 4.10). *Let R be a commutative noetherian ring and \mathfrak{c}_R be as above. Then $\text{cosupp}_R R = \mathcal{V}(\mathfrak{c}_R)$ if and only if R/\mathfrak{c}_R has full cosupport.*

For example, if R is a ring such that Theorem 1 applies to R/\mathfrak{c}_R , then equality of (\star) holds. However, Example 5.6 shows that strict inequality in (\star) can occur, providing a negative answer to a question in [SWW17]; moreover, this example shows that $\text{cosupp}_R R$ need not be a closed subset of $\text{Spec}(R)$, unlike the support of a finitely generated module.

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We briefly outline the paper: In the first section, we set notation and define cosupport for our setting. In Section 2, we show how cosupport can be detected by minimal complexes of cotorsion flat modules; the main result here is Theorem 2.5. In the next section, we compute cosupport of cotorsion modules. In Section 4, we compare cosupport and support of finitely generated modules (Corollary 4.3), prove a lemma describing how cosupport passes along finite ring maps (Lemma 4.4), and establish Theorem 1 from above. Finally, in Section 5 we give a number of explicit examples of cosupport of commutative noetherian rings, including an example that exhibits a ring without closed cosupport (Example 5.6).

1. COSUPPORT IN A COMMUTATIVE NOETHERIAN RING

Throughout this paper, all rings are commutative noetherian. We briefly set notation, discuss certain derived functors, and define cosupport.

1.1. Setting and notation. Let R be a commutative noetherian ring. Our main objects of study are complexes of R -modules, primarily in the derived category $D(R)$. Briefly, a *complex* of R -modules is a \mathbb{Z} -graded R -module along with a differential whose square is zero. A complex C , whose differential is understood to be ∂_C , is written as

$$\dots \xrightarrow{\partial_C^{i-1}} C^i \xrightarrow{\partial_C^i} C^{i+1} \xrightarrow{\partial_C^{i+1}} \dots,$$

where we primarily index cohomologically. We say that a complex C is *bounded on the left (respectively, right)* if $C^i = 0$ for $i \ll 0$ (respectively, $C^i = 0$ for $i \gg 0$). For complexes C, D , the tensor product complex $C \otimes_R D$ and Hom complex $\text{Hom}_R(C, D)$ are defined as direct sum and direct product totalizations of their corresponding double complexes, respectively.

The *homotopy category* $K(R)$ is the category whose objects are complexes of R -modules and morphisms are chain maps up to chain homotopy. If we also invert all quasi-isomorphisms between complexes (chain maps that induce an isomorphism on cohomology), we obtain the (*unbounded*) *derived category* of R , denoted $D(R)$. We use \simeq to denote isomorphisms in $D(R)$ (i.e., to indicate that two complexes are quasi-isomorphic). For details on complexes, homotopies, and the derived category, see for example [Avr98] or [Wei94, Chapter 10].

We say a complex F is *semiflat* if F^i is flat for $i \in \mathbb{Z}$ and $F \otimes_R -$ preserves quasi-isomorphisms. A complex P is *semiprojective* if P^i is projective for $i \in \mathbb{Z}$ and $\text{Hom}_R(P, -)$ preserves quasi-isomorphisms. Dually, a complex I is *semiinjective* if I^i is injective for $i \in \mathbb{Z}$ and $\text{Hom}_R(-, I)$ preserves quasi-isomorphisms. (These are the same as the “DG-flat/projective/injective” complexes of [AF91].) Every complex $M \in D(R)$ has a *semiprojective resolution* (and hence also a *semiflat resolution*) $F \xrightarrow{\simeq} M$ (for existence of such resolutions, see [Spa88, Proposition 5.6] and also [AF91, 1.6]); similarly, semiinjective resolutions exist. This extends the classical notions of projective, flat, and injective resolutions of modules.

1.2. Derived completion and colocalization. We remind the reader of two functors on $D(R)$ that will be used to define cosupport: left-derived completion and right-derived colocalization. The \mathfrak{p} -adic completion of a module M is $\Lambda^{\mathfrak{p}} M = \varprojlim_n (R/\mathfrak{p}^n \otimes_R M)$. This extends to a functor on the homotopy category $\Lambda^{\mathfrak{p}} : K(R) \rightarrow K(R)$ and has a left derived functor $\mathbf{L}\Lambda^{\mathfrak{p}} : D(R) \rightarrow D(R)$, defined using semiprojective resolutions; see [AJL97] and [PSY14]. For any complex $M \in D(R)$, and semiflat resolution $F \xrightarrow{\simeq} M$ (or, more generally, a semiflat complex F with $F \simeq M$), we have

$$(1.1) \quad \mathbf{L}\Lambda^{\mathfrak{p}} M \simeq \varprojlim_n (R/\mathfrak{p}^n \otimes_R F)$$

in $D(R)$ [Lip02, page 31]; see also [PSY14, Proposition 3.6]. The functor $\mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, -) : D(R) \rightarrow D(R)$, referred to as right-derived colocalization, is the usual right derived functor of $\text{Hom}_R(R_{\mathfrak{p}}, -)$; if $M \xrightarrow{\simeq} I$ is a semiinjective resolution of M , then

$$\mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, M) \simeq \text{Hom}_R(R_{\mathfrak{p}}, I).$$

1.3. Cosupport. For any complex $M \in D(R)$, we define (as in [BIK12]) the *cosupport* of M to be

$$(1.2) \quad \text{cosupp}_R M = \{\mathfrak{p} \in \text{Spec}(R) \mid H^* \mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, \mathbf{L}\Lambda^{\mathfrak{p}} M) \neq 0\}.$$

This agrees with a variety of other ways to define cosupport; for example, setting $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$, the residue field of $R_{\mathfrak{p}}$, we have [SWW17, Proposition 4.4]:

$$(1.3) \quad \mathfrak{p} \in \text{cosupp}_R M \iff \mathbf{R}\text{Hom}_R(\kappa(\mathfrak{p}), M) \not\cong 0 \iff \kappa(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}}^{\mathbf{L}} \mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, M) \not\cong 0.$$

In particular, for a flat R -module F , the cosupport of F is just

$$(1.4) \quad \text{cosupp}_R F = \{\mathfrak{p} \in \text{Spec}(R) \mid \text{Ext}_R^*(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} F) \neq 0\}.$$

For comparison, the (*small*) *support* of a complex M with $H^i(M) = 0$ for $i \ll 0$ is:

$$\text{supp}_R M = \{\mathfrak{p} \in \text{Spec}(R) \mid H_*(\kappa(\mathfrak{p}) \otimes_R^{\mathbf{L}} M) \neq 0\}.$$

Equivalently, $\text{supp}_R M$ is the set of prime ideals such that $E(R/\mathfrak{p})$ occurs as one of the indecomposable injective modules in the minimal semiinjective resolution of M [Fox79, Remark 2.9], see also [CI10]; this motivates the main result in the next section.

2. COSUPPORT VIA MINIMAL COMPLEXES OF COTORSION FLAT MODULES

An R -module T is called *cotorsion flat* if it is both flat and satisfies $\text{Ext}_R^1(F, T) = 0$ for every flat module F (i.e., it is also cotorsion). Enochs showed [Eno84, Theorem] that cotorsion flat modules decompose uniquely as a product of completions of free $R_{\mathfrak{q}}$ -modules, for $\mathfrak{q} \in \text{Spec}(R)$; namely, T is cotorsion flat if and only if

$$(2.1) \quad T \cong \prod_{\mathfrak{q} \in \text{Spec}(R)} \widehat{R_{\mathfrak{q}}^{(X_{\mathfrak{q}})}}^{\mathfrak{q}},$$

for some (possibly empty or infinite) sets $X_{\mathfrak{q}}$. Set $T_{\mathfrak{q}} = \widehat{R_{\mathfrak{q}}^{(X_{\mathfrak{q}})}}^{\mathfrak{q}}$. For a cotorsion flat module T as in (2.1) and a fixed prime \mathfrak{p} , we have [Tho17, Lemma 2.2]:

$$(2.2) \quad \Lambda^{\mathfrak{p}}(T) \cong \prod_{\mathfrak{q} \supseteq \mathfrak{p}} T_{\mathfrak{q}} \quad \text{and} \quad \text{Hom}_R(R_{\mathfrak{p}}, T) \cong \prod_{\mathfrak{q} \subseteq \mathfrak{p}} T_{\mathfrak{q}}.$$

Furthermore, for any complex B of cotorsion flat modules, the natural maps $R \rightarrow R_{\mathfrak{p}}$ and $R \rightarrow \Lambda^{\mathfrak{p}} R$ induce degree-wise split maps: $\text{Hom}_R(R_{\mathfrak{p}}, B) \hookrightarrow B$ and $B \twoheadrightarrow \Lambda^{\mathfrak{p}} B$.

2.1. Minimal complexes of cotorsion flat modules. As in [AM02], we say a complex C is *minimal* if every homotopy equivalence $\gamma : C \rightarrow C$ is an isomorphism. Similar to minimality criteria for injective resolutions or projective resolutions of finitely generated modules in a local ring, we have a minimality criterion for complexes of cotorsion flat modules:

Theorem 2.3. [Tho17, Theorem 3.5] *For a commutative noetherian ring R , a complex B of cotorsion flat modules is minimal if and only if the complex $R/\mathfrak{p} \otimes_R \text{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B)$ has zero differential for every $\mathfrak{p} \in \text{Spec}(R)$.*

2.2. Cotorsion flat resolutions and replacements. For an R -module M , a *right cotorsion flat resolution* of M is a complex B of cotorsion flat modules along with a quasi-isomorphism $M \xrightarrow{\sim} B$ such that $B^i = 0$ for $i < 0$. Dually, a *left cotorsion flat resolution* of M is a complex B of cotorsion flat modules with $B^i = 0$ for $i > 0$ together with a quasi-isomorphism $B \xrightarrow{\sim} M$.

Minimal right cotorsion flat resolutions exist for flat modules and minimal left cotorsion flat resolutions exist for cotorsion modules [Tho17, Proposition 4.8]. Moreover, if R has finite Krull dimension, there exists [Tho17, Proposition 4.9] a diagram of quasi-isomorphisms

$$(2.4) \quad B \xleftarrow{\simeq} F \xrightarrow{\simeq} M$$

where F is a left flat resolution of M (in fact, built from flat covers) and B is a semiflat, minimal complex of cotorsion flat modules; we call such a diagram a *minimal cotorsion flat replacement* of M .

2.3. Detecting cosupport. We now show that a minimal cotorsion flat replacement of a module can be used to detect its cosupport, dual to the fact that minimal injective resolutions detect support [Fox79]. This will be a primary tool in computing cosupport in the remainder of this paper. For a complex B of cotorsion flat modules, we colloquially say \mathfrak{p} *appears in* B if $\widehat{R_{\mathfrak{p}}^{(X_{\mathfrak{p}})} }^{\mathfrak{p}}$ is a nonzero summand of B^i for some i . If R is a commutative noetherian ring of finite Krull dimension, M is an R -module, and B is a minimal cotorsion flat replacement of M , then the following result shows that $\mathfrak{p} \in \text{cosupp}_R(M)$ if and only if \mathfrak{p} appears in B .

Theorem 2.5. *Let R be a commutative noetherian ring and M be a complex in $\mathbf{D}(R)$. If there exists a semiflat, minimal complex of cotorsion flat modules B such that $B \simeq M$, and one of the following holds:*

- (i) $\text{pd}_R R_{\mathfrak{p}} < \infty$ for every $\mathfrak{p} \in \text{Spec}(R)$, or
- (ii) B is bounded on the left, that is, $B^i = 0$ for all $i \ll 0$,

then

$$\mathfrak{p} \in \text{cosupp}_R(M) \iff \widehat{R_{\mathfrak{p}}^{(X)}}^{\mathfrak{p}} \neq 0 \text{ is a summand of } B^i, \text{ some } i \in \mathbb{Z} \text{ and set } X.$$

Remark 2.6. If R has finite Krull dimension, then work of Jensen [Jen70, Proposition 6] and Raynaud-Gruson [RG71, Seconde partie, Théorème 3.2.6] implies that R satisfies condition (i) of the theorem, and we need no boundedness assumptions on B . On the other hand, the minimal right cotorsion flat resolution of any flat module is semiflat and bounded on the left, and so to compute $\text{cosupp}_R R$, we need not impose any additional finiteness conditions on R .

Proof of Theorem 2.5. Since B is semiflat and $B \simeq M$, we have that $\mathfrak{p} \in \text{cosupp}_R(M)$ if and only if $\mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B) \not\cong 0$ by (1.1) and definition (1.2). We will show that $\mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B) \simeq \text{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B)$ and that $\text{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B) \not\cong 0$ if and only if $\widehat{R_{\mathfrak{p}}^{(X)}}^{\mathfrak{p}}$ is a non-zero direct summand of B^i , for some $i \in \mathbb{Z}$ and some set X .

By (2.2), the complex $\text{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B)$ can be identified with the subquotient complex

$$\cdots \rightarrow \widehat{R_{\mathfrak{p}}^{(X_{\mathfrak{p}}^i)}}^{\mathfrak{p}} \rightarrow \widehat{R_{\mathfrak{p}}^{(X_{\mathfrak{p}}^{i+1})}}^{\mathfrak{p}} \rightarrow \cdots$$

of B . We then have:

$$\begin{aligned}
\widehat{R_{\mathfrak{p}}^{(X)}}^{\mathfrak{p}} & \text{ is a non-zero summand of } B^i, \text{ for some } i \in \mathbb{Z} \text{ and set } X, \\
& \iff \mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B) \neq 0 \\
& \iff R/\mathfrak{p} \otimes_R \mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B) \neq 0, \text{ as } R/\mathfrak{p} \otimes_R \widehat{R_{\mathfrak{p}}^{(X)}}^{\mathfrak{p}} \cong (R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})^{(X)}, \\
& \iff R/\mathfrak{p} \otimes_R \mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B) \not\cong 0, \text{ by minimality of } B \text{ and Theorem 2.3.}
\end{aligned}$$

It remains to show that $R/\mathfrak{p} \otimes_R \mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B) \not\cong 0$ if and only if $\mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B) \neq 0$ and that $\mathbf{R}\mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B) \simeq \mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B)$.

Claim 1: The complex $\mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B)$ is acyclic if and only if $R/\mathfrak{p} \otimes_R \mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B)$ is acyclic.

Proof of Claim 1: Suppose $\mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B)$ is acyclic and set $K^i = \ker(T_{\mathfrak{p}}^i \rightarrow T_{\mathfrak{p}}^{i+1})$. For each $i \in \mathbb{Z}$, the short exact sequence $0 \rightarrow K^i \rightarrow T_{\mathfrak{p}}^i \rightarrow K^{i+1} \rightarrow 0$, along with minimality of B (using Theorem 2.3), induces an exact sequence

$$R/\mathfrak{p} \otimes_R K^i \rightarrow R/\mathfrak{p} \otimes_R T_{\mathfrak{p}}^i \xrightarrow{0} R/\mathfrak{p} \otimes_R K^{i+1} \rightarrow 0.$$

It follows that $R/\mathfrak{p} \otimes_R K^i = 0$ for every $i \in \mathbb{Z}$, and hence $R/\mathfrak{p} \otimes_R T_{\mathfrak{p}}^i = 0 = T_{\mathfrak{p}}^i$. Therefore $\mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B) = 0$, and the forward implication follows.

Conversely, minimality of B implies that $R/\mathfrak{p} \otimes_R \mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B)$ has zero differential by Theorem 2.3, and so acyclicity of this complex implies it is in fact the zero complex, hence $\mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B) = 0$ also.

Claim 2: Assuming either condition (i) or (ii), there is a quasi-isomorphism:

$$\mathbf{R}\mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B) \simeq \mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B).$$

Proof of Claim 2: Let $f : P \xrightarrow{\sim} R_{\mathfrak{p}}$ be a projective resolution (chosen to be bounded if condition (i) holds) and set $C = \mathrm{cone}(f)$. The triangulated functor $\mathrm{Hom}_R(-, \Lambda^{\mathfrak{p}} B)$ on $\mathbf{K}(R)$ yields an exact triangle

$$(2.7) \quad \mathrm{Hom}_R(C, \Lambda^{\mathfrak{p}} B) \rightarrow \mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B) \xrightarrow{f^*} \mathrm{Hom}_R(P, \Lambda^{\mathfrak{p}} B) \rightarrow .$$

Since C is an acyclic complex of flat R -modules that is bounded on the right, all kernels of C are also flat, hence $\mathrm{Hom}_R(C, (\Lambda^{\mathfrak{p}} B)^i)$ is acyclic for each i . If either condition (i) or (ii) holds, it follows by [CFH06, Lemma 2.5] that $\mathrm{Hom}_R(C, \Lambda^{\mathfrak{p}} B)$ is acyclic. The five lemma applied to (2.7) gives that f^* is a quasi-isomorphism. Since $\mathbf{R}\mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B) \simeq \mathrm{Hom}_R(P, \Lambda^{\mathfrak{p}} B)$ in $\mathbf{D}(R)$, this verifies Claim 2. \square

An immediate consequence of Theorem 2.5 is that we are able to easily construct a module with a prescribed cosupport.

Corollary 2.8. *Let $W \subseteq \mathrm{Spec}(R)$ be any subset. Then $M = \prod_{\mathfrak{p} \in W} \widehat{R_{\mathfrak{p}}}^{\mathfrak{p}}$ is an R -module with $\mathrm{cosupp}_R(M) = W$.*

3. COSUPPORT OF COTORSION MODULES

The purpose of this section is to illustrate how minimal cotorsion flat resolutions can be utilized to compute the cosupport of a cotorsion module, since every cotorsion module has such a resolution [Tho17, Corollary 4.8].

For a ring R and prime $\mathfrak{p} \in \mathrm{Spec}(R)$, the module $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is cotorsion, since $\kappa(\mathfrak{p}) \cong \mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), E(R/\mathfrak{p}))$.

Proposition 3.1. *Let R be a commutative ring of finite Krull dimension. Then*

$$\operatorname{cosupp}_R \kappa(\mathfrak{p}) = \{\mathfrak{p}\} = \operatorname{supp}_R \kappa(\mathfrak{p}).$$

Proof. As $\operatorname{supp}_R \kappa(\mathfrak{p}) = \{\mathfrak{p}\}$, the minimal injective resolution $\kappa(\mathfrak{p}) \xrightarrow{\sim} I$ involves only $E(R/\mathfrak{p})$. Thus:

$$\operatorname{Hom}_R(I, E(R/\mathfrak{p})) \xrightarrow{\sim} \operatorname{Hom}_R(\kappa(\mathfrak{p}), E(R/\mathfrak{p})) \cong \kappa(\mathfrak{p}).$$

Since \mathfrak{p} is the only prime appearing in I , we obtain:

$$\operatorname{Hom}_R(I, E(R/\mathfrak{p})) = \cdots \rightarrow \widehat{R_{\mathfrak{p}}^{(X_{\mathfrak{p}}^1)}} \rightarrow \widehat{R_{\mathfrak{p}}^{(X_{\mathfrak{p}}^0)}} \rightarrow 0$$

is a left cotorsion flat resolution of $\kappa(\mathfrak{p})$ with $X_{\mathfrak{q}}^i = 0$ for all $\mathfrak{q} \neq \mathfrak{p}$ and $X_{\mathfrak{p}}^0 \neq 0$. We claim $\operatorname{Hom}_R(I, E(R/\mathfrak{p}))$ is minimal: Since I is minimal, $\operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), I_{\mathfrak{p}})$ has zero differential, and therefore:

$$\begin{aligned} R/\mathfrak{p} \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, \operatorname{Hom}_R(I, E(R/\mathfrak{p}))) &\cong \kappa(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} \operatorname{Hom}_{R_{\mathfrak{p}}}(I_{\mathfrak{p}}, E(R/\mathfrak{p})) \\ &\cong \operatorname{Hom}_{R_{\mathfrak{p}}}(\operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), I_{\mathfrak{p}}), E(R/\mathfrak{p})) \end{aligned}$$

has zero differential as well, implying that $\operatorname{Hom}_R(I, E(R/\mathfrak{p}))$ is minimal by Theorem 2.3. Finally, the result follows by Theorem 2.5. \square

The cosupport of an injective module has been described elsewhere, see [BIK12, Proposition 5.4] and [SWW17, Proposition 6.3], but we give a different proof here to illustrate the use of minimal complexes of cotorsion flat modules to compute cosupport.

Proposition 3.2. *Let R be a commutative noetherian ring of finite Krull dimension and $E(R/\mathfrak{p})$ an indecomposable injective R -module. Then*

$$\operatorname{cosupp}_R E(R/\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec}(R) \mid \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Proof. Let $R \xrightarrow{\sim} I$ be the minimal injective resolution of R . Then we have a quasi-isomorphism

$$\operatorname{Hom}_R(I, E(R/\mathfrak{p})) \xrightarrow{\sim} \operatorname{Hom}_R(R, E(R/\mathfrak{p})) \cong E(R/\mathfrak{p}).$$

Since I is minimal, the complex $\operatorname{Hom}_R(R/\mathfrak{q}, I_{\mathfrak{q}}) (\cong \operatorname{Hom}_{R_{\mathfrak{q}}}(\kappa(\mathfrak{q}), I_{\mathfrak{q}}))$ has zero differential for all $\mathfrak{q} \in \operatorname{Spec}(R)$, and thus

$$\begin{aligned} R/\mathfrak{q} \otimes_R \operatorname{Hom}_R(R_{\mathfrak{q}}, \operatorname{Hom}_R(I, E(R/\mathfrak{p}))) &\cong R/\mathfrak{q} \otimes_R \operatorname{Hom}_R(I_{\mathfrak{q}}, E(R/\mathfrak{p})) \\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{q}, I_{\mathfrak{q}}), E(R/\mathfrak{p})) \end{aligned}$$

has zero differential as well. Hence $\operatorname{Hom}_R(I, E(R/\mathfrak{p}))$ is a minimal left cotorsion flat resolution by Theorem 2.3. There is an isomorphism:

$$\operatorname{Hom}_R(E(R/\mathfrak{q}), E(R/\mathfrak{p})) \cong \begin{cases} \widehat{R_{\mathfrak{q}}^{(X_{\mathfrak{q}})}}^{\mathfrak{q}}, & \text{for some set } X_{\mathfrak{q}} \neq \emptyset, \quad \mathfrak{q} \subseteq \mathfrak{p} \\ 0 & \mathfrak{q} \not\subseteq \mathfrak{p} \end{cases}.$$

For $\mathfrak{q} \not\subseteq \mathfrak{p}$, this is well-known; for $\mathfrak{q} \subseteq \mathfrak{p}$, one uses adjointness to show that

$$\operatorname{Hom}_R(E(R/\mathfrak{q}), E(R/\mathfrak{p})) \cong \operatorname{Hom}_R(E(R/\mathfrak{q}), \operatorname{Hom}_R(R_{\mathfrak{q}}, E(R/\mathfrak{p}))),$$

and the fact that $\operatorname{Hom}_R(R_{\mathfrak{q}}, E(R/\mathfrak{p}))$ is an injective $R_{\mathfrak{q}}$ -module. Finally, apply [Xu96, Lemma 4.1.5]. The result follows from Theorem 2.5, using that for $\mathfrak{q} \subseteq \mathfrak{p}$, $E(R/\mathfrak{q})$ appears in I , hence there exists $X_{\mathfrak{q}}^i \neq \emptyset$ for some $i \in \mathbb{Z}$ in $\operatorname{Hom}_R(I, E(R/\mathfrak{p}))$. \square

4. COSUPPORT OF FINITELY GENERATED MODULES

We turn our focus to computing cosupport of finitely generated modules (or complexes with degreewise finitely generated cohomology). In order to do so, we will employ the following fact about $\text{cosupp}_R X \otimes_R^{\mathbf{L}} Y$, which complements the corresponding fact [BIK12, Theorem 9.7] that $\text{cosupp}_R \mathbf{R}\text{Hom}_R(X, Y) = \text{supp}_R X \cap \text{cosupp}_R Y$ for any complexes $X, Y \in \mathbf{D}(R)$.

Proposition 4.1. *Let R be a commutative noetherian ring and $X, Y \in \mathbf{D}(R)$ be complexes. Suppose one of the following holds:*

- (1) $\text{pd}_R R_{\mathfrak{p}} < \infty$ for every $\mathfrak{p} \in \text{Spec}(R)$, $H^i(X)$ is finitely generated for each i , and Y is a bounded complex of flat modules, or
- (2) $\text{pd}_R R_{\mathfrak{p}} < \infty$ for every $\mathfrak{p} \in \text{Spec}(R)$, $H^i(X)$ is finitely generated for each i , $H^i(X) = 0$ for $i \gg 0$, and Y has bounded cohomology, or
- (3) X is a bounded complex of finitely generated projectives and Y has bounded cohomology.

Then

$$\text{cosupp}_R X \otimes_R^{\mathbf{L}} Y = \text{supp}_R X \cap \text{cosupp}_R Y.$$

Proof. Fix $\mathfrak{p} \in \text{Spec}(R)$ and consider the natural tensor evaluation map

$$(4.2) \quad \omega : \mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, Y) \otimes_R^{\mathbf{L}} X \rightarrow \mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, Y \otimes_R^{\mathbf{L}} X).$$

We claim that ω is a quasi-isomorphism under each of the conditions (1), (2), or (3) above. Assuming (2) or (3) holds, ω is an isomorphism by [CH09, Proposition 2.2(vi,v)]. If (1) holds, choose a bounded projective resolution $P \xrightarrow{\sim} R_{\mathfrak{p}}$ and consider the functors $F = \mathbf{R}\text{Hom}_R(P, Y) \otimes_R^{\mathbf{L}} -$ and $G = \mathbf{R}\text{Hom}_R(P, Y \otimes_R^{\mathbf{L}} -)$ on $\mathbf{D}(R)$ with natural transformation $\eta : F \rightarrow G$ given by the natural map above. The map $\eta(M)$ is a quasi-isomorphism for any finitely generated module M by (2). As Y is a bounded complex of flat modules, F and G are way-out functors in the sense of [Har66, I, section 7]. It follows that (4.2) is a quasi-isomorphism for all complexes X with finitely generated cohomology by [Har66, I, Proposition 7.1].

We have that $\mathfrak{p} \in \text{cosupp}_R X \otimes_R^{\mathbf{L}} Y$ if and only if the following hold, using the equivalent descriptions of cosupport in (1.3):

$$\begin{aligned} & \mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, Y \otimes_R^{\mathbf{L}} X) \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p}) \neq 0 \\ & \iff \mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, Y) \otimes_R^{\mathbf{L}} X \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p}) \neq 0, \text{ by (4.2),} \\ & \iff (\mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, Y) \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} (X \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p})) \neq 0 \\ & \iff \mathfrak{p} \in \text{cosupp}_R Y \cap \text{supp}_R X, \end{aligned}$$

where the last equivalence follows from the Künneth formula [Wei94, Theorem 3.6.3]. \square

Part (1) of the following corollary extends [SWW17, Theorem 6.6] to unbounded complexes and some rings without dualizing complexes, including all rings of finite Krull dimension. Recall that an R -complex is called *perfect* if it is quasi-isomorphic to a bounded complex of finitely generated projective R -modules.

Corollary 4.3. *Let X be an R -complex with degreewise finitely generated cohomology. If at least one of the following holds:*

- (1) $\text{pd}_R R_{\mathfrak{p}} < \infty$ for every $\mathfrak{p} \in \text{Spec}(R)$,
- (2) $X \in \mathbf{D}(R)$ is a perfect complex,

then

$$\operatorname{cosupp}_R X = \operatorname{supp}_R X \cap \operatorname{cosupp}_R R.$$

Proof. Apply the previous proposition with $Y = R$. \square

The corollary puts emphasis on computing the cosupport of R . Recall [Tho17, Corollary 4.8] that R has a minimal right cotorsion flat resolution; indeed, the *minimal pure-injective resolution*¹ of R (built from pure-injective envelopes; see [EJ00] for details) is such a resolution. This allows us to invoke Enochs' description [Eno87] of minimal pure-injective resolutions in order to determine $\operatorname{cosupp}_R R$.

The following change of rings lemma allows us to compare cosupport along finite ring maps, by understanding cotorsion flat modules under finite base change (cf. [BIK12, Theorem 7.11]). A ring homomorphism $R \rightarrow S$ is called *finite* if S is finitely generated as an R -module; in addition, to every ring map $f : R \rightarrow S$ we can associate a map $f^* : \operatorname{Spec}(S) \rightarrow \operatorname{Spec}(R)$ defined by sending a prime ideal $\mathfrak{q} \subseteq S$ to its contraction $f^{-1}(\mathfrak{q}) \subseteq R$, which is necessarily prime as well.

Lemma 4.4. *Let $f : R \rightarrow S$ be a finite map of commutative noetherian rings. Then*

$$\left(\prod_{\mathfrak{p} \in \operatorname{Spec}(R)} \widehat{R_{\mathfrak{p}}^{(X_{\mathfrak{p}})} }^{\mathfrak{p}} \right) \otimes_R S \cong \prod_{\mathfrak{q} \in \operatorname{Spec}(S)} \widehat{S_{\mathfrak{q}}^{(X_{\mathfrak{q}})}}^{\mathfrak{q}}, \text{ where } X_{\mathfrak{q}} = X_{\mathfrak{p}} \text{ for } f^*(\mathfrak{q}) = \mathfrak{p}.$$

Consequently,

$$\operatorname{cosupp}_S S = (f^*)^{-1}(\operatorname{cosupp}_R R),$$

or in other words, for $\mathfrak{q} \in \operatorname{Spec}(S)$, $\mathfrak{q} \in \operatorname{cosupp}_S S$ if and only if $f^*(\mathfrak{q}) \in \operatorname{cosupp}_R R$.

Proof. For a prime $\mathfrak{p} \in \operatorname{Spec}(R)$, the set $(f^*)^{-1}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec}(S) \mid f^*(\mathfrak{q}) = \mathfrak{p}\}$ is finite. Fix $\mathfrak{p} \in \operatorname{Spec}(R)$ and set $W = (f^*)^{-1}(\mathfrak{p})$. We will show

$$(4.5) \quad \widehat{R_{\mathfrak{p}}^{(X_{\mathfrak{p}})}}^{\mathfrak{p}} \otimes_R S \cong \bigoplus_{\mathfrak{q} \in W} \widehat{S_{\mathfrak{q}}^{(X_{\mathfrak{p}})}}^{\mathfrak{q}}.$$

This is enough to establish the first claim, as S is finitely generated as an R -module. The assertion regarding cosupport then follows from Theorem 2.5 applied to a minimal pure-injective resolution of R , as follows: Let $R \xrightarrow{\sim} B$ be a minimal pure-injective resolution of R (i.e., a right resolution built from pure-injective envelopes). Applying $-\otimes_R S$ preserves pure-injective envelopes because S is finitely generated as an R -module, so that $S \xrightarrow{\sim} B \otimes_R S$ is a minimal pure-injective resolution of S [Eno87, Theorem 4.2]. By [Tho17, Corollary 4.8], B and $B \otimes_R S$ are minimal right cotorsion flat resolutions of R and S , respectively. By Theorem 2.5, the primes appearing in B are precisely those in $\operatorname{cosupp}_R R$ and the primes appearing in $B \otimes_R S$ are those in $\operatorname{cosupp}_S S$. The statement relating the cosupport of R and S now follows once we have verified (4.5).

To establish (4.5), we recall the following fact [Rah09, Theorem 1.1]:

$$(4.6) \quad \operatorname{Hom}_R(S, E_R(R/\mathfrak{p})) \cong \bigoplus_{\mathfrak{q} \in W} E_S(S/\mathfrak{q}).$$

¹Minimal pure-injective resolutions were referred to as right \mathcal{PI} -resolutions in [Tho17].

With this in hand, we apply $-\otimes_R S$ to the cotorsion flat module $\widehat{R_{\mathfrak{p}}^{(X_{\mathfrak{p}})}}^{\mathfrak{p}}$:

$$\begin{aligned} \widehat{R_{\mathfrak{p}}^{(X_{\mathfrak{p}})}}^{\mathfrak{p}} \otimes_R S &\cong \operatorname{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p})^{(X_{\mathfrak{p}})}) \otimes_R S, \text{ by [Xu96, Lemma 4.1.5]}, \\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(S, E(R/\mathfrak{p})), E(R/\mathfrak{p})^{(X_{\mathfrak{p}})}), \text{ using that } S \text{ is f.g.}, \\ &\cong \operatorname{Hom}_S(\operatorname{Hom}_R(S, E(R/\mathfrak{p})), \operatorname{Hom}_R(S, E(R/\mathfrak{p}))^{(X_{\mathfrak{p}})}), \text{ adjoints, } S \text{ f.g.}, \\ &\cong \bigoplus_{\mathfrak{q} \in W} \operatorname{Hom}_S(E_S(S/\mathfrak{q}), E_S(S/\mathfrak{q})^{(X_{\mathfrak{p}})}), \text{ by (4.6) and the remarks below,} \\ &\cong \bigoplus_{\mathfrak{q} \in W} \widehat{S_{\mathfrak{q}}^{(X_{\mathfrak{p}})}}^{\mathfrak{q}}, \text{ by [Xu96, Lemma 4.1.5]}, \end{aligned}$$

where the second to last isomorphism follows because, for $\mathfrak{q}', \mathfrak{q}'' \in W$, if $\mathfrak{q}' \subseteq \mathfrak{q}''$, then $\mathfrak{q}' = \mathfrak{q}''$ [AM69, Corollary 5.9] and if $\mathfrak{q}' \not\subseteq \mathfrak{q}''$, then $E(R/\mathfrak{q}'')^{(X_{\mathfrak{p}})}$ is \mathfrak{q}' -local [Rot09, Theorem 2.65] and so $\operatorname{Hom}_R(E(R/\mathfrak{q}'), E(R/\mathfrak{q}'')^{(X_{\mathfrak{p}})}) = 0$. \square

Remark 4.7. An immediate consequence for a finite ring map $R \rightarrow S$ is that if $\operatorname{cosupp}_R R = \operatorname{Spec}(R)$, then $\operatorname{cosupp}_S S = \operatorname{Spec}(S)$. If the map $\pi : R \twoheadrightarrow S$ is surjective, then for $\mathfrak{p} \in \mathcal{V}(\ker(\pi)) \subseteq \operatorname{Spec}(R)$, we have $\pi(\mathfrak{p}) \in \operatorname{cosupp}_S S$ if and only if $\mathfrak{p} \in \operatorname{cosupp}_R R$.

Remark 4.8. This lemma recovers the fact that if R is a commutative noetherian ring and \mathfrak{m} is a maximal ideal, then $\mathfrak{m} \in \operatorname{cosupp}_R R$.² From the finite map $\pi : R \twoheadrightarrow R/\mathfrak{m}$, we see that since $0 \in \operatorname{cosupp}_{R/\mathfrak{m}} R/\mathfrak{m}$ and $\pi^*(0) = \mathfrak{m}$, that $\mathfrak{m} \in \operatorname{cosupp}_R R$.

Recall that for any commutative noetherian ring R we have³ the following inclusion [BIK12, Proposition 4.19]:

$$(4.9) \quad \operatorname{cosupp}_R R \subseteq \bigcap_{R \text{ is } \mathfrak{a}\text{-complete}} \mathcal{V}(\mathfrak{a}),$$

where $\mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$. Set $\mathfrak{c}_R = \sum \mathfrak{a}$, with the sum over all ideals \mathfrak{a} such that R is \mathfrak{a} -complete. Note that R is \mathfrak{c}_R -complete and if $\mathfrak{b} \supsetneq \mathfrak{c}_R$, then R is not \mathfrak{b} -complete. Also, $\bigcap \mathcal{V}(\mathfrak{a}) = \mathcal{V}(\mathfrak{c}_R)$, where the intersection is taken over all ideals \mathfrak{a} such that R is \mathfrak{a} -complete.

One of our goals is to investigate when the inclusion $\operatorname{cosupp}_R R \subseteq \mathcal{V}(\mathfrak{c}_R)$ is an equality; in particular, we show that equality holds for any ring R such that R/\mathfrak{c}_R is countable (using Theorem 4.11 below).

Proposition 4.10. *Let R be a commutative noetherian ring and let \mathfrak{c}_R be defined as above. The following are equivalent:*

- (1) *Equality in (4.9) holds; i.e., $\operatorname{cosupp}_R R = \mathcal{V}(\mathfrak{c}_R)$;*
- (2) *R/\mathfrak{c}_R has full cosupport; i.e., $\operatorname{cosupp}_{R/\mathfrak{c}_R}(R/\mathfrak{c}_R) = \operatorname{Spec}(R/\mathfrak{c}_R)$;*
- (3) *For every $\mathfrak{p} \in \mathcal{V}(\mathfrak{c}_R)$, $\operatorname{Ext}_{R/\mathfrak{p}}^i((R/\mathfrak{p})_{(0)}, R/\mathfrak{p}) \neq 0$ for some i .*

Proof. For any ideal $I \subseteq R$ and $\mathfrak{p} \in \mathcal{V}(I)$, we have $\mathfrak{p}/I \in \operatorname{cosupp}_{R/I} R/I$ if and only if $\mathfrak{p} \in \operatorname{cosupp}_R R$; see Remark (4.7). In conjunction with the inclusion (4.9), the equivalence of (1) and (2) then follows for $I = \mathfrak{c}_R$. Moreover, for $\mathfrak{p} \in \mathcal{V}(\mathfrak{c}_R)$,

²In fact, $\max(\operatorname{cosupp}_R M) = \max(\operatorname{supp}_R M)$ for any $M \in \mathcal{D}(R)$ [BIK12, Theorem 4.13].

³This fact can also be gleaned from the minimal pure-injective resolution of R , using also [Eno95, Corollary 2.6].

$\mathfrak{p} \in \text{cosupp}_R R$ if and only if $0 \in \text{cosupp}_{R/\mathfrak{p}} R/\mathfrak{p}$, and hence (1) is equivalent to (3) by definition (1.4). \square

We caution that equality in (4.9) need not hold in general; see Example 5.6 below. Indeed, equality need not hold even for noetherian domains of finite Krull dimension that are only 0-complete (i.e., not complete at any nonzero ideal).

Part (3) of the following result gives an affirmative answer to the first part of [SWW17, Question 6.16] and part (2) avoids the assumption of a dualizing complex of [SWW17, Theorem 6.11]. This result also displays some of the subtleties of cosupport; indeed, there are rings of any Krull dimension having full cosupport, see part (1), and also rings of any cardinality having full cosupport, see part (3). Part (2), along with Corollary 4.12 below, recovers [BIK12, Proposition 4.18] and [SWW17, Theorem 6.11].

Theorem 4.11. *If R is one of the following rings, then $\text{cosupp}_R R = \text{Spec}(R)$.*

- (1) *A countable commutative noetherian ring;*
- (2) *A 1-dimensional commutative noetherian domain, not complete local;*
- (3) *The ring $k[x, y]$, for any field k .*

Moreover, if $R \rightarrow T$ is a finite ring map, then T also satisfies $\text{cosupp}_T T = \text{Spec}(T)$.

Proof. For finite ring maps $R \rightarrow T$, Lemma 4.4 shows that if $\text{cosupp}_R R = \text{Spec}(R)$, then $\text{cosupp}_T T = \text{Spec}(T)$.

To address (1), let R be any countable commutative noetherian ring. For $\mathfrak{p} \in \text{Spec}(R)$, Lemma 4.4 shows that $\mathfrak{p} \in \text{cosupp}_R R$ if and only if $0 \in \text{cosupp}_{R/\mathfrak{p}} R/\mathfrak{p}$. Therefore, it is sufficient to assume R is a countable domain and show $0 \in \text{cosupp}_R R$.

If R is a field, R trivially has full cosupport. It is therefore enough to consider the case where R is not a field, in which case there exists a short exact sequence (see [Tho15, (3.1)])

$$0 \rightarrow R \rightarrow \varprojlim_{s \in S} R/sR \rightarrow \text{Ext}_R^1(R_{(0)}, R) \rightarrow 0,$$

where $S = R \setminus \{0\}$ is pre-ordered by divisibility: $s'|s$ if and only if $sR \subseteq s'R$. In this case, $\varprojlim_{s \in S} R/sR$ is uncountable. Since R is countable, the first map in this short exact sequence is thus not surjective. Hence $\text{Ext}_R^1(R_{(0)}, R) \neq 0$. By definition (1.4), $0 \in \text{cosupp}_R R$. It follows that rings as in (1) have full cosupport.

Next, if R is as in (2), then since $\dim(R) = 1$, the minimal pure-injective resolution of R has the form [Eno87]:

$$B := 0 \rightarrow \prod_{\mathfrak{m} \text{ maximal}} \widehat{R}^{\mathfrak{m}} \rightarrow T_0 \rightarrow 0.$$

As R is a domain that is not complete local, we must have $T_0 \neq 0$. Since B is a semiflat, minimal right cotorsion flat resolution of R [Tho17, Lemma 4.8], Theorem 2.5 yields that $\text{cosupp}_R R = \text{Spec}(R)$.

For (3), if k is countable, the result follows from (1), so assume k is uncountable. In this case, the ring $R = k[x, y]$ satisfies $\text{Ext}_R^2(R_{(0)}, R) \neq 0$ by [Gru71, Proposition 3.2]. Thus $0 \in \text{cosupp}_R R$. For $0 \neq \mathfrak{p} \in \text{Spec}(R)$, the ring R/\mathfrak{p} is either a field or a ring as in (2), and so has full cosupport. Applying Lemma 4.4 to the map $R \rightarrow R/\mathfrak{p}$ for each $\mathfrak{p} \neq 0$, we obtain that $\text{cosupp}_R R = \text{Spec}(R)$. \square

We conclude that cosupport and support coincide for any complex with degree-wise finitely generated cohomology over any of the rings in Theorem 4.11.

Corollary 4.12. *Let R be any ring as in Theorem 4.11, and let M be a complex of R -modules with degree-wise finitely generated cohomology. Then*

$$\operatorname{cosupp}_R(M) = \operatorname{supp}_R(M).$$

Proof. By Corollary 4.3 and Theorem 4.11, we have

$$\operatorname{cosupp}_R(M) = \operatorname{cosupp}_R(R) \cap \operatorname{supp}_R(M) = \operatorname{supp}_R(M).$$

□

Corollary 4.13. *Let R be a commutative noetherian ring. If R/\mathfrak{c}_R is countable, then*

$$\operatorname{cosupp}_R R = \mathcal{V}(\mathfrak{c}_R).$$

Proof. Combine Proposition 4.10 and Theorem 4.11. □

A conjecture, initiated in the early 1970s by Gruson [Gru71] and Jensen [Jen72], and then generalized by Gruson in 2013 and formalized by Thorup [Tho15], allows us to conjecture that rings having full cosupport are far more ubiquitous. The conjecture of Gruson-Jensen states the following: For a field k and integer $n \geq 0$, let $R = k[x_1, \dots, x_n]$ be the polynomial ring in n variables. Set $c = 0$ if k is finite and define c by the cardinality $|k| = \aleph_c$ if k is infinite. With this setup, it is conjectured that:

$$(4.14) \quad \operatorname{Ext}_R^i(R_{(0)}, R) \neq 0 \iff i = \inf\{c + 1, n\}.$$

This conjecture is true when k is at most countable or $n \leq 1$; e.g., see [Tho15]. If the Gruson-Jensen conjecture were true, it would follow from Noether normalization that finite type rings over any field would have all minimal primes in their cosupport; an inductive argument on the dimension of finite type rings over a field, along with Lemma 4.4, would imply that finite type rings over any field would have full cosupport. Moreover, if R is any commutative noetherian ring such that for any minimal prime \mathfrak{p} over \mathfrak{c}_R , the domain R/\mathfrak{p} is a finite type k -algebra, this conjecture would imply such a ring satisfies $\operatorname{cosupp}_R R = \mathcal{V}(\mathfrak{c}_R)$.

5. EXAMPLES OF $\operatorname{cosupp}_R R$

The following question is motivated by Proposition 4.10.

Question 5.1. *When do noetherian domains that are only 0-complete have full cosupport?*

The examples below illustrate the nuances of this question. We start with a warm-up of some examples of rings having full cosupport.

Example 5.2. The following rings R satisfy $\operatorname{cosupp}_R R = \operatorname{Spec}(R)$:

- (1) Let k be a countable field and $R = k[x_1, \dots, x_n]/\mathfrak{a}$, for $n \geq 1$ and an ideal \mathfrak{a} ;
- (2) Let k be a (possibly uncountable) field and $R = k[x_1, x_2]/\mathfrak{a}$, for an ideal \mathfrak{a} ;
- (3) Let R be Nagata's example [Nag62, Appendix, Example 1] of a commutative noetherian ring of infinite Krull dimension, under the additional assumption that the coefficient field is countable;
- (4) Let $R = \mathbb{Z}_{(p)}$ for any prime number p ; or more generally, any non-complete discrete valuation ring.

These all follow immediately from Theorem 4.11: (1) and (3) are countable, (2) is a finite ring extension of $k[x, y]$ for an arbitrary field, and (4) is dimension 1 and not complete local.

For contrast, recall that the cosupport of R can fall short of $\text{Spec}(R)$; in particular, the cosupport of a complete semi-local ring is the set of maximal ideals (cf. [BIK12, Proposition 4.19]):

Example 5.3. For a complete semi-local ring R with maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$, the minimal right cotorsion flat resolution has one term: $\prod_{i=1}^n \widehat{R_{\mathfrak{m}_i}}^{\mathfrak{m}_i}$. Theorem 2.5 implies that $\text{cosupp}_R(R) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$. In particular, a complete local ring (R, \mathfrak{m}) has cosupport equal to $\{\mathfrak{m}\}$.

In order to understand how cosupport behaves under adjoining power series variables, we first prove:

Proposition 5.4. *Let S be a ring that is I -complete. Then the canonical surjection $\pi : S \twoheadrightarrow S/I$ induces a homeomorphism of topological spaces:*

$$\pi^* : \text{cosupp}_{S/I}(S/I) \xrightarrow{\cong} \text{cosupp}_S S.$$

Proof. The natural surjection $\pi : S \twoheadrightarrow S/I$ induces a homeomorphism of topological spaces (i.e., a continuous bijection whose inverse is also continuous) $\pi^* : \text{Spec}(S/I) \xrightarrow{\cong} \mathcal{V}(I) \subseteq \text{Spec}(S)$ [AM69, Chapter 1, Exercise 21]. Since S is I -complete, $\text{cosupp}_S S \subseteq \mathcal{V}(I)$. For $\mathfrak{p} \in \text{Spec}(S/I)$, Lemma 4.4 implies that $\mathfrak{p} \in \text{cosupp}_{S/I}(S/I)$ if and only if $\pi^*(\mathfrak{p}) \in \text{cosupp}_S S$. Hence π^* induces a bijection between $\text{cosupp}_{S/I}(S/I)$ and $\text{cosupp}_S S$. Indeed, endowing $\text{cosupp}_{S/I}(S/I) \subseteq \text{Spec}(S/I)$ and $\text{cosupp}_S S \subseteq \mathcal{V}(I)$ each with the subspace topology, we obtain that $\pi^* : \text{cosupp}_{S/I}(S/I) \xrightarrow{\cong} \text{cosupp}_S S$ is a homeomorphism. \square

Example 5.5. If R is any ring and $S = R[[t_1, \dots, t_n]]$ for $n \geq 0$, then Proposition 5.4 yields a homeomorphism

$$\text{cosupp}_R R \xrightarrow{\cong} \text{cosupp}_S S,$$

using that S is (t_1, \dots, t_n) -complete [Mat89, Exercise 8.6]. In particular, if $S = k[[x]][[t]]$, then $\text{cosupp}_S S = \mathcal{V}((t))$.

The next example we give shows that the cosupport of R need not be a closed subset of $\text{Spec}(R)$, i.e., there are rings R such that $\text{cosupp}_R R \neq \mathcal{V}(I)$ for any ideal I . In particular, it shows that we can have a strict inequality $\text{cosupp}_R R \subsetneq \mathcal{V}(\mathfrak{c}_R)$. This provides a negative answer to the question [SWW17, Question 6.13].

Example 5.6. Let k be a field and set $T = k[[t]][x]$. Applying Lemma 4.4 to the finite map $T \twoheadrightarrow T/(x) \cong k[[t]]$, the fact that $0 \notin \text{cosupp}_{k[[t]]} k[[t]]$ (see Example 5.3) implies that $(x) \notin \text{cosupp}_T T$, so that $\text{cosupp}_T T \subsetneq \mathcal{V}((0))$.

The ring T has uncountably many height 1 prime ideals that are maximal [HRW06, Theorem 3.1, Remarks 3.2], even if k is finite. Let \mathcal{P} be the set of all height 1 maximal ideals⁴. As $\mathfrak{p} \in \mathcal{P}$ are maximal, Remark 4.8 implies that $\mathfrak{p} \in \text{cosupp}_T T$.

⁴For our purposes, we only need \mathcal{P} to be an infinite set, and we may take $\mathcal{P} = \{(1 - xt^n)\}_{n \geq 1}$. To see that for each $n \geq 1$, $\mathfrak{p}_n := (1 - xt^n)$ is a maximal ideal, just observe that every nonzero element of T/\mathfrak{p}_n is a unit; this follows because the images of x and t are both units.

Since T is a noetherian unique factorization domain, every height 1 prime ideal is principal [Mat89, Theorem 20.1]. For each $\mathfrak{p} \in \mathcal{P}$, we may write $\mathfrak{p} = (f_{\mathfrak{p}})$, for a prime element $f_{\mathfrak{p}} \in T$. Define $I = \bigcap_{\mathfrak{p} \in \mathcal{P}} (f_{\mathfrak{p}})$, which is an ideal. If $\alpha \in I$ is an element, then α must be divisible by $f_{\mathfrak{p}}$ for every $\mathfrak{p} \in \mathcal{P}$, forcing $\alpha = 0$ since T is a unique factorization domain. Therefore $I = 0$. If $\text{cosupp}_T T \subseteq \mathcal{V}(J)$ for some ideal $J \subseteq T$, then $\mathfrak{p} \supseteq J$ for every $\mathfrak{p} \in \mathcal{P}$, hence $0 = I \supseteq J$, i.e., $J = 0$. However, $\text{cosupp}_T T \neq \mathcal{V}((0))$, hence it is not a closed subset of $\text{Spec}(T)$ and we have a strict containment $\text{cosupp}_R R \subsetneq \mathcal{V}(\mathfrak{c}_R)$.

Example 5.7. Let k be any field and set $S = k[[t]][x][[s_1, \dots, s_n]]$, for $n \geq 0$. We claim that $\text{cosupp}_S S$ is not a closed subset of $\text{Spec}(S)$. There is a canonical surjection $\pi : S \twoheadrightarrow T$, where T is as in Example 5.6, which induces a homeomorphism $\pi^* : \text{Spec}(T) \xrightarrow{\cong} \mathcal{V}((s_1, \dots, s_n))$. As S is (s_1, \dots, s_n) -complete, $\text{cosupp}_S S \subseteq \mathcal{V}((s_1, \dots, s_n))$, and so Proposition 5.4 shows that $(\pi^*)^{-1}(\text{cosupp}_S S) = \text{cosupp}_T T$. As π^* is continuous and $\text{cosupp}_T T$ is not closed, $\text{cosupp}_S S$ cannot be closed in $\text{Spec}(S)$. This yields a family of rings without closed cosupport.

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